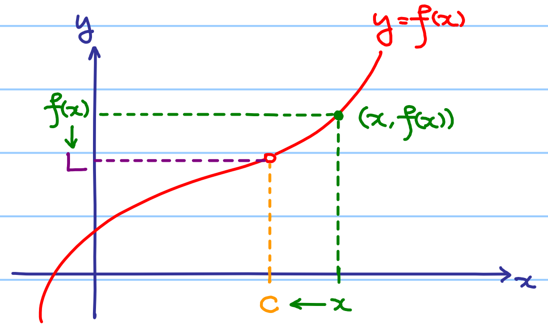
§6 Limits of Functions

ε-δ Definition

Recall : Single variable calculus

Definition 6.1 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer^{+} to c from both sides, then L is called the limit of $f(x)$ at c, and we write $\lim_{x\to c} f(x) = L$.

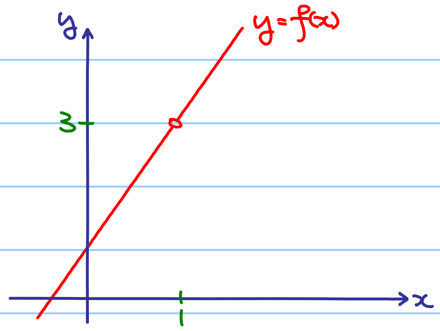


Remark: $\lim_{x\to c} f(x)$ studies the behavior of f "near" $x=c$ but not the the behavior of f exactly at $x=c$

Example 6.1

Let $D = \mathbb{R}\setminus\{1\}$ and let $f: D \to \mathbb{R}$ defined by $f(x) = \frac{2x^2-x-1}{x-1}$.

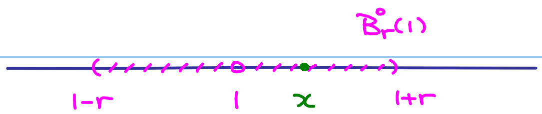
$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{2x^2-x-1}{x-1} = \lim_{x\to 1} 2x+1 = 3$



Remark:

$x=1$ is not in the domain D of f, but we can still discuss $\lim_{x\to 1} f(x)$

The main reason is that for all $r>0$, $B_r^{\circ}(1) \cap D \neq \phi$, i.e. 1 is a cluster point of D.



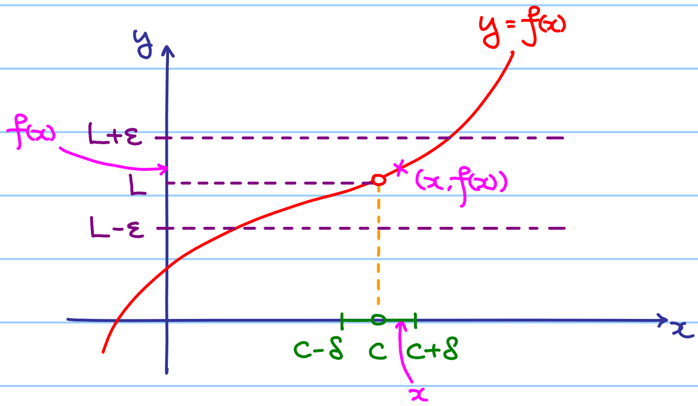
x is "close" to 1 and $f(x)$ is well-defined

Definition 6.2 (ε-δ definition)

Let $D \subseteq \mathbb{R}$, c be a cluster point of D and $f: D \to \mathbb{R}$ be a function.

$L \in \mathbb{R}$ is said to be the limit of f at the point c, and we write $\lim_{x \to c} f(x) = L$, if

for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in D$ with $0 < |x - c| < \delta$.



Meaning: No matter how small ε is given,

we can always find $\delta > 0$ such that if $x \in D$ with $\underbrace{0 <}_{\text{i.e. } x \neq c} \text{dist}(x, c) < \delta$,

then $f(x)$ lies in the ε-tunnel (ε-neighborhood of L)

Example 6.1 (Continue)

Let $D = \mathbb{R} \setminus \{1\}$ and let $f: D \to \mathbb{R}$ defined by $f(x) = \frac{2x^2 - x - 1}{x - 1}$.

Prove that $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3$ by ε-δ definition.

Given $\varepsilon > 0$, how to choose $\delta > 0$ such that

$|f(x) - 3| < \varepsilon$ for all $x \in D$ with $0 < |x - 1| < \delta$?

$\Updownarrow$

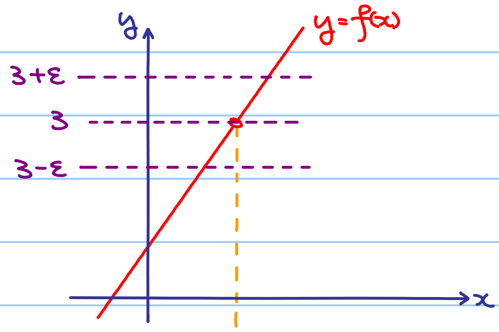
$|(2x+1) - 3| < \varepsilon$ $\quad$ ($\because f(x) = 2x + 1$ when $x \neq 1$)

$\Updownarrow$

$|2x - 2| < \varepsilon$

$\Updownarrow$

$|x - 1| < \frac{\varepsilon}{2}$ $\rightsquigarrow$ choose $\delta = \frac{\varepsilon}{2}$!!



proof:

Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2} > 0$, then for all $x \in D$ with $0 < |x - 1| < \delta = \frac{\varepsilon}{2}$,

we have $|2x - 2| < \varepsilon$

$$|(2x+1) - 3| < \varepsilon$$

$$\left| \frac{2x^2 - x - 1}{x - 1} - 3 \right| < \varepsilon$$

$$|f(x) - 3| < \varepsilon$$

$\therefore \lim_{x \to 1} f(x) = 3$

Naturally, we can extend definition 6.2 to higher dimensional cases:

Definition 6.3 (ε-δ definition)

Let $D \subseteq \mathbb{R}^n$, $\vec{c}$ be a cluster point of D and $f: D \to \mathbb{R}$ be a function.

$L \in \mathbb{R}$ is said to be the limit of f at the point $\vec{c}$, and we write $\lim_{\vec{x} \to \vec{c}} f(\vec{x}) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(\vec{x}) - L| < \varepsilon$ for all $\vec{x} \in D$ with $0 < |\vec{x} - \vec{c}| < \delta$.

Remark: If $D \subseteq \mathbb{R}^n$ is open, by exercise 4.2, every point $\vec{c} \in D$ is a cluster point of D and so it makes sense to ask whether $\lim_{\vec{x} \to \vec{c}} f(\vec{x})$ exists or not.

Example 6.2

Let $f(x,y) = \dfrac{4xy^2}{x^2+y^2}$. Show that $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

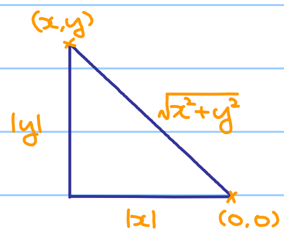
Remark: The maximum domain of $f = D = \mathbb{R}^2 \setminus \{(0,0)\}$ and $(0,0)$ is a cluster point of D, so it makes sense to discuss $\lim_{(x,y) \to (0,0)} f(x,y)$.

Given $\varepsilon > 0$, how to choose $\delta > 0$ such that

$|f(x,y) - 0| < \varepsilon$ for all $x \in D$ with $0 < |(x,y) - (0,0)| < \delta$?

$$\Big| \frac{4xy^2}{x^2+y^2} - 0 \Big| < \varepsilon \qquad\qquad 0 < \sqrt{x^2+y^2} < \delta$$

$$\frac{4|x| \, y^2}{x^2+y^2} < \varepsilon, \qquad \frac{y^2}{x^2+y^2} \le 1$$



Great! Only need to control $|x|$,

but it can be done since $|x| < \sqrt{x^2+y^2} < \delta$

proof:

Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{4} > 0$, then for all $x \in D$ with $0 < |(x,y) - (0,0)| < \delta$,

we have $|x| < \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{4}$ and $\dfrac{y^2}{x^2+y^2} \le 1$. Then,

$$|f(x,y) - 0| = \Big| \frac{4xy^2}{x^2+y^2} - 0 \Big| = 4 \cdot |x| \cdot \frac{y^2}{x^2+y^2} < 4 \cdot \frac{\varepsilon}{4} \cdot 1 = \varepsilon$$

$\therefore \lim_{(x,y) \to (0,0)} f(x,y) = 0.$

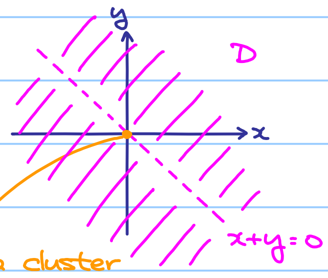
Example 6.3

Let $f(x,y) = \dfrac{x^2-y^2-x-y}{x+y}$.

The maximum domain of $f = D = \mathbb{R}^2 \setminus \{(x,y)\in\mathbb{R}^2 : x+y=0\}$

and $(0,0)$ is a cluster point of D,

so it makes sense to discuss $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.



$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2-x-y}{x+y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y-1)}{x+y}$$

$$= \lim_{(x,y)\to(0,0)} x-y-1$$

$$= -1$$

Remark: Some may write $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x+y\neq 0}} \frac{x^2-y^2-x-y}{x+y}$.

Exercise 6.1

By using δ-ε definition, show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2-x-y}{x+y} = -1$.

proof:

Let $D = \{(x,y)\in\mathbb{R}^2 . x+y\neq 0\}$.

Given $\varepsilon>0$, take $\delta = \frac{\varepsilon}{2}>0$, then for all $x\in D$ with $0<|(x,y)-(0,0)|<\delta$,

we have $|x| < \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{2}$ and $|y| < \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{2}$

$$|f(x,y)-(-1)| = \left|\frac{x^2-y^2-x-y}{x+y} - (-1)\right| = |x-y| \le |x|+|y| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

Note: if $(x,y)\in D$, $x+y\neq 0$

$\therefore \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2-x-y}{x+y} = -1$.

Exercise 6.2

Let $f(x,y) = \dfrac{x-y}{\sqrt{x}+\sqrt{y}}$.

a) Find the maximum domain D of f and hence show that $(0,0)$ is a cluster point of D.

b) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

Definition 6.4 (ε-δ definition)

Let $D \subseteq \mathbb{R}^n$, $\vec{c}$ be a cluster point of D and $f: D \to \mathbb{R}^m$ be a function.

$\vec{L} \in \mathbb{R}^m$ is said to be the limit of f at the point $\vec{c}$, and we write $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = \vec{L}$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(\vec{x}) - \vec{L}| < \varepsilon$ for all $\vec{x} \in D$ with $0 < |\vec{x} - \vec{c}| < \delta$.

Think: Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be a function and we write $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$

where $f_1, f_2: \mathbb{R}^3 \to \mathbb{R}$ are two real valued functions.

Question 1: $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f_1(x,y,z) = L_1$ and $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f_2(x,y,z) = L_2 \overset{?}{\Rightarrow} \lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = (L_1, L_2)$

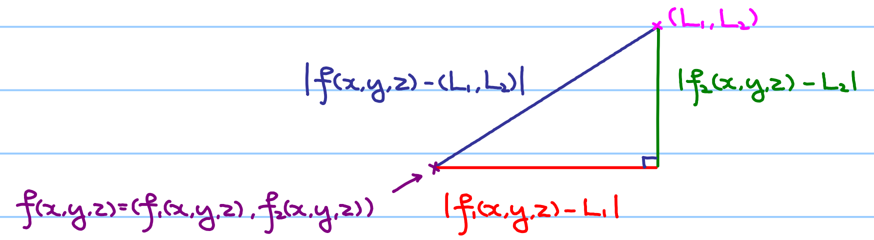
Question 2: $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = (L_1, L_2) \overset{?}{\Rightarrow} \lim_{(x,y,z)\to(x_0,y_0,z_0)} f_1(x,y,z) = L_1$ and $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f_2(x,y,z) = L_2$

Answer: Yes, for both!

The reason is quite simple, note that $|f(x,y,z) - (L_1,L_2)|^2 = |f_1(x,y,z) - L_1|^2 + |f_2(x,y,z) - L_2|^2$,

as (x,y,z) tends to (x_0,y_0,z_0),

$|f(x,y,z) - (L_1,L_2)|$ is getting smaller $\Leftrightarrow$ both $|f_1(x,y,z) - L_1|$ and $|f_2(x,y,z) - L_2|$ are getting smaller



In general, if $D \subseteq \mathbb{R}^n$, $\vec{c}$ be a cluster point of D and $f: D \to \mathbb{R}^m$ be a function.

we write $f(\vec{x}) = f(x_1, x_2, \cdots, x_n) = (f_1(x_1,x_2,\cdots,x_n), f_2(x_1,x_2,\cdots,x_n), \cdots, f_m(x_1,x_2,\cdots,x_n))$, we have:

Proposition 6.1

$\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = \vec{L} = (L_1, L_2, \cdots, L_m) \in \mathbb{R}^m$ if and only if $\lim_{\vec{x}\to\vec{c}} f_i(\vec{x}) = L_i$ for all $i = 1, 2, \cdots, m$.

Consequence. It is good enough to focus on real valued functions.

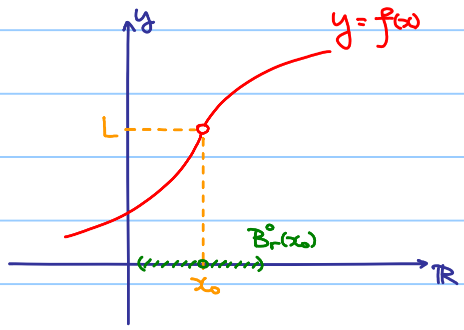
Example 6.4

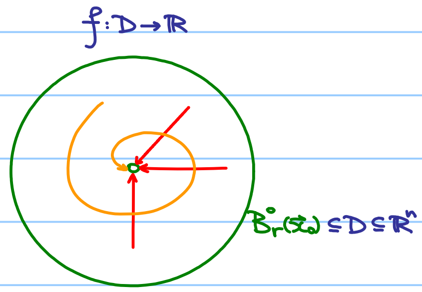
Evaluate $\lim_{(x,y,z)\to(0,0,0)} (e^{x+y+z}, 2+x+y+z)$.

Note that $\lim_{(x,y,z)\to(0,0,0)} e^{x+y+z} = 1$ and $\lim_{(x,y,z)\to(0,0,0)} 2+x+y+z = 2$.

$\therefore \lim_{(x,y,z)\to(0,0,0)} (e^{x+y+z}, 2+x+y+z) = (\lim_{(x,y,z)\to(0,0,0)} e^{x+y+z}, \lim_{(x,y,z)\to(0,0,0)} 2+x+y+z) = (1, 2)$

Non-existence of Limits





For one dimensional case, x can get closer to x_0 from left and right hand side.

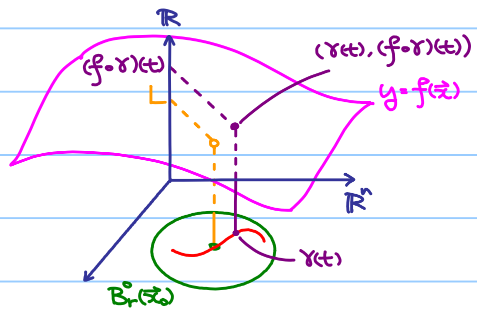
However, for higher dimensional case, there are many choices

An analogue to the fact $\lim\limits_{x\to x_0} f(x) = L \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = \lim\limits_{x\to x_0^+} f(x) = L$ in one dimension, we have

Proposition 6.2

Let $D \subseteq \mathbb{R}^n$ and let $\vec{x}_0 \in \mathbb{R}^n$ such that $B_r^\circ(\vec{x}_0) \subseteq D$ for some $r>0$ and let $f: D \to \mathbb{R}$.

$\lim\limits_{x\to x_0} f(x) = L$ if and only if for any curve $\gamma: (-\varepsilon, \varepsilon) \to B_r(\vec{x}_0)$ with $\gamma(0) = \vec{x}_0$, we have $\lim\limits_{t\to 0} (f\circ\gamma)(t) = L$.



Example 6.5

Let $f(x,y) = \dfrac{x^4 - y^4 + x^2 + y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4-y^4+x^2+y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2+1)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x^2-y^2+1 = 1.$$

Let $f(x,y) = \dfrac{x^4 - y^4 + x^2 + y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$ We have $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$

Let $\gamma_1(t) = (t,0)$, $t \in \mathbb{R}$. Then $\gamma_1(0) = (0,0)$ and

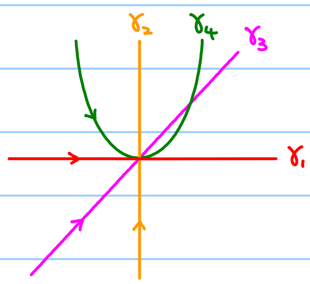
$$\lim_{t\to 0} (f\circ\gamma_1)(t) = \lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{t^4+t^2}{t^2} = \lim_{t\to 0} 1+t^2 = 1.$$

Let $\gamma_2(t) = (0,t)$, $t\in\mathbb{R}$. Then $\gamma_2(0) = (0,0)$ and

$$\lim_{t\to 0} (f\circ\gamma_2)(t) = \lim_{t\to 0} f(\gamma_2(t)) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{-t^4+t^2}{t^2} = \lim_{t\to 0} 1-t^2 = 1.$$

Let $\gamma_3(t) = (t,t)$, $t\in\mathbb{R}$. Then $\gamma_3(0) = (0,0)$ and

$$\lim_{t\to 0} (f\circ\gamma_3)(t) = \lim_{t\to 0} f(\gamma_3(t)) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t^4-t^4+t^2+t^2}{t^2+t^2} = 1.$$



Exercise: Let $\gamma_4(t) = (t, t^2)$, $t\in\mathbb{R}$. Check $\gamma_4(0) = (0,0)$ and see if $\lim\limits_{t\to 0} (f\circ\gamma_4)(t) = 1$.

In general, it would be difficult to use proposition 6.2 to prove $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})$ exists since there are infinitely many simple curves to be checked.

However, we can use it to prove a limit does not exist

Recall the 1-dimensional case, by showing that either

1) $\lim_{x\to x_0^+} f(x)$ does not exist or $\lim_{x\to x_0^-} f(x)$ does not exist;

2) Both $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ exist, but $\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x)$,

we know $\lim_{x\to x_0} f(x)$ does not exist.

The analogue in higher dimensional case, by showing that either

1) There exists a curve γ with $\gamma(0)=\vec{x}_0$ but $\lim_{t\to 0}(f\circ\gamma)(t)$ does not exist;

2) There exist curves γ_1 and γ_2 with $\gamma_1(0)=\gamma_2(0)=\vec{x}_0$ but $\lim_{t\to 0}(f\circ\gamma_1)(t) \neq \lim_{t\to 0}(f\circ\gamma_2)(t)$,

we know $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})$ does not exist.

Remark: This result can be extended suitably to functions with various domains.

Example 6.6

Let $f(x,y)=\sin\left(\frac{1+x}{x^2+y^2}\right)$, $(x,y)\neq(0,0)$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

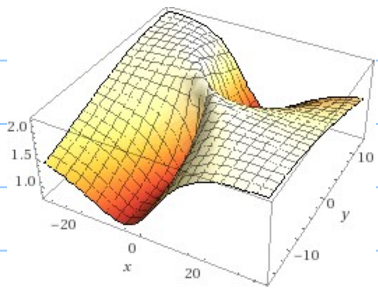
Let $\gamma(t)=(0,t)$ Then $\gamma(0)=(0,0)$, but $\lim_{t\to 0}(f\circ\gamma)(t)=\lim_{t\to 0} f(\gamma(t))=\lim_{t\to 0} f(0,t)=\lim_{t\to 0}\sin\left(\frac{1}{t^2}\right)$ does NOT exist.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 6.7

Let $f(x,y)=\frac{2x^2-xy+y^2}{x^2+y^2}$, $(x,y)\neq(0,0)$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



Let $\gamma_1(t)=(t,0)$ and $\gamma_2(t)=(0,t)$ Then, $\gamma_1(0)=\gamma_2(0)=(0,0)$, but

$\lim_{t\to 0}(f\circ\gamma_1)(t)=\lim_{t\to 0} f(\gamma_1(t))=\lim_{t\to 0} f(t,0)=\lim_{t\to 0}\frac{2t^2}{t^2}=\lim_{t\to 0} 2=2$

$\lim_{t\to 0}(f\circ\gamma_2)(t)=\lim_{t\to 0} f(\gamma_2(t))=\lim_{t\to 0} f(0,t)=\lim_{t\to 0}\frac{t^2}{t^2}=\lim_{t\to 0} 1=1$

$\therefore \lim_{t\to 0}(f\circ\gamma_1)(t)\neq\lim_{t\to 0}(f\circ\gamma_2)(t)$ and $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Exercise 6.3

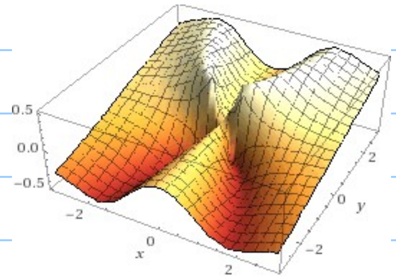
Let $f(x,y)=\frac{x+y}{x-y}$.

a) Find the maximum domain D of f and show that $(0,0)$ is a cluster point of D.

b) Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 6.8

Let $f(x,y)=\frac{x^2y}{x^4+y^2}$, $(x,y)\neq(0,0)$.



Let $\gamma_1(t)=(t,mt)$, where $m\in\mathbb{R}$ and $\gamma_2(t)=(0,t)$.

Then, $\gamma_1(0)=\gamma_2(0)=(0,0)$, but

$$\lim_{t\to0}(f\circ\gamma_1)(t)=\lim_{t\to0}f(\gamma_1(t))=\lim_{t\to0}f(t,mt)=\lim_{t\to0}\frac{mt^3}{t^4+m^2t^2}=\lim_{t\to0}\frac{mt}{t^2+m^2}=0$$

((x,y) tends to $(0,0)$ along the straight line $y=mx$, in particular it is the x-axis when $m=0$.)

$$\lim_{t\to0}(f\circ\gamma_2)(t)=\lim_{t\to0}f(\gamma_2(t))=\lim_{t\to0}f(0,t)=\lim_{t\to0}\frac{0}{t^2}=\lim_{t\to0}0=0$$

((x,y) tends to $(0,0)$ along the y-axis.)

As (x,y) tends to $(0,0)$ along any straight line, $f(x,y)$ tends to 0.

However, it is NOT enough to conclude $\lim_{(x,y)\to(0,0)}f(x,y)$ exists.

Let $\gamma_3(t)=(t,t^2)$. Then $\gamma(0)=(0,0)$, and

$$\lim_{t\to0}(f\circ\gamma_3)(t)=\lim_{t\to0}f(\gamma_3(t))=\lim_{t\to0}f(t,t^2)=\lim_{t\to0}\frac{t^4}{t^4+t^4}=\lim_{t\to0}\frac{1}{2}=\frac{1}{2}$$

((x,y) tends to $(0,0)$ along the parabola $y=x^2$.)

As (x,y) tends to $(0,0)$ along the parabola $y=x^2$, $f(x,y)$ tends to $\frac{1}{2}$.

$\therefore \lim_{t\to0}(f\circ\gamma_2)(t)\neq\lim_{t\to0}(f\circ\gamma_3)(t)$ and $\lim_{(x,y)\to(0,0)}f(x,y)$ does not exist

Finding Limits Using Polar Coordinates

Recall:

Let $D\subseteq\mathbb{R}^2$, $\vec{0}=(0,0)$ be a cluster point of D and $f:D\to\mathbb{R}$ be a function.

$L\in\mathbb{R}$ is said to be the limit of f at $\vec{0}=(0,0)$, and we write $\lim_{(x,y)\to(0,0)}f(x,y)=L$, if

for all $\varepsilon>0$, there exists $\delta>0$ such that $|f(x,y)-L|<\varepsilon$ for all $\vec{x}\in D$ with $0<|(x,y)-(0,0)|<\delta$

$$\downarrow \qquad\qquad \downarrow$$

By using polar coordinates: $|f(r\cos\theta, r\sin\theta)-L|<\varepsilon$ $\qquad$ $0<|r-0|<\delta$

Proposition 6.3

$\lim_{(x,y)\to(0,0)}f(x,y)=L$ if and only if $\lim_{r\to0}f(r\cos\theta,r\sin\theta)=L$ for all $\theta\in\mathbb{R}$.

Example 6.9

Find $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0.$$

Example 6.10

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta + r^2\sin\theta\cos\theta}{r^2} = \lim_{r\to 0} \cos^2\theta + \sin\theta\cos\theta$$ which depends on θ $\left(= \begin{cases} 1 & \text{if } \theta = 0 \\ 0 & \text{if } \theta = \frac{\pi}{2} \end{cases} \right)$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2+xy}{x^2+y^2}$ does not exist.

Algebraic Properties

Proposition 6.4 (Algebraic Properties)

If $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = L$, $\lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = M$, where $L, M \in \mathbb{R}$, then

1) $\lim_{\vec{x}\to\vec{x}_0} (f(\vec{x}) + g(\vec{x})) = \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) + \lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = L + M$

2) $\lim_{\vec{x}\to\vec{x}_0} (f(\vec{x}) - g(\vec{x})) = \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) - \lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = L - M$

3) $\lim_{\vec{x}\to\vec{x}_0} (f(\vec{x}) g(\vec{x})) = (\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}))(\lim_{\vec{x}\to\vec{x}_0} g(\vec{x})) = LM$

4) If $\lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = M \neq 0$, $\lim_{\vec{x}\to\vec{x}_0} \left(\frac{f(\vec{x})}{g(\vec{x})}\right) = \frac{\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})}{\lim_{\vec{x}\to\vec{x}_0} g(\vec{x})} = \frac{L}{M}$

Example 6.11

Intuitively, as (x,y) tends to $(0,0)$, both x and y are "small" and so $\lim_{(x,y)\to(0,0)} \frac{x+1}{xy-2} = -\frac{1}{2}$.

Can we prove it without using ε-δ definition "painfully"?

Exercise: Using ε-δ definition to show that $\lim_{(x,y)\to(0,0)} x = 0$, $\lim_{(x,y)\to(0,0)} y = 0$ and $\lim_{(x,y)\to(0,0)} c = c$ for all $c \in \mathbb{R}$.

$$\lim_{(x,y)\to(0,0)} x = 0, \quad \lim_{(x,y)\to(0,0)} 1 = 1 \overset{①}{\Rightarrow} \lim_{(x,y)\to(0,0)} x+1 = 1$$

$$\lim_{(x,y)\to(0,0)} x = 0, \quad \lim_{(x,y)\to(0,0)} y = 0 \overset{③}{\Rightarrow} \lim_{(x,y)\to(0,0)} xy = 0$$

$$\lim_{(x,y)\to(0,0)} xy = 0, \quad \lim_{(x,y)\to(0,0)} 2 = 2 \overset{②}{\Rightarrow} \lim_{(x,y)\to(0,0)} xy-2 = -2$$

$$\lim_{(x,y)\to(0,0)} xy-2 = -2, \quad \lim_{(x,y)\to(0,0)} x+1 = 1 \overset{④}{\Rightarrow} \lim_{(x,y)\to(0,0)} \frac{x+1}{xy-2} = -\frac{1}{2}$$

Sandwich Theorem

Theorem 6.1 (Sandwich Theorem)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \mathbb{R}^n$ be a cluster point of D and $f, g, h : D \to \mathbb{R}$ are three functions.

If there exists $r>0$ such that $g(\vec{x}) \le f(\vec{x}) \le h(\vec{x})$ for all $\vec{x} \in B_r^{\circ}(\vec{x}_0) \cap D$ and $\lim_{\vec{x}\to\vec{x}_0} g(x) = \lim_{\vec{x}\to\vec{x}_0} h(x) = L$,

then $\lim_{\vec{x}\to\vec{x}_0} f(x) = L$.

Idea: Estimate a function $f(\vec{x})$ by some functions $g(\vec{x})$ and $h(\vec{x})$ that we understand well.

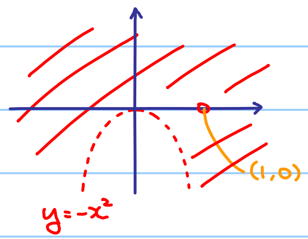
Example 6.12

Show that $\lim_{(x,y)\to(1,0)} \ln(x^2+y) \sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right) = 0$.

Note: $\ln(x^2+y)$ is defined when $x^2+y>0$

$\sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right)$ is defined when $(x,y) \ne (1,0)$

The shaded region is the maximum domain of $\ln(x^2+y)\sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right)$.

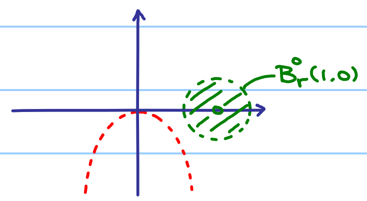


$$-1 \le \sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right) \le 1 \quad \text{for all } (x,y) \ne (1,0)$$

$$-|\ln(x^2+y)| \le \ln(x^2+y) \le |\ln(x^2+y)| \quad \text{for all } (x,y) \text{ with } x^2+y>0$$

$$-|\ln(x^2+y)| \le \ln(x^2+y)\sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right) \le |\ln(x^2+y)| \quad \text{for all } (x,y) \in B_r^{\circ}(1,0) \text{ for some small } r>0$$

$$\lim_{(x,y)\to(1,0)} -|\ln(x^2+y)| = \lim_{(x,y)\to(1,0)} |\ln(x^2+y)| = 0$$



$\therefore$ By sandwich theorem, $\lim_{(x,y)\to(1,0)} \ln(x^2+y)\sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right) = 0$

Remark: Don't write $\lim_{(x,y)\to(1,0)} \ln(x^2+y)\sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right)$

$$= \left(\lim_{(x,y)\to(1,0)} \ln(x^2+y)\right)\cdot\left(\lim_{(x,y)\to(1,0)} \sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right)\right) = 0\cdot\left(\lim_{(x,y)\to(1,0)} \sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right)\right) = 0$$

since $\lim_{(x,y)\to(1,0)} \sin\left(\frac{1}{\sqrt{(x-1)^2+y^2}}\right)$ does not exist.

Iterated Limits

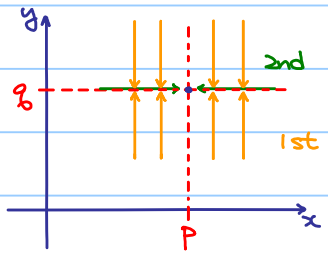
① $\lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right)$ means taking limit with $y\to q$ first, followed by taking limit with $x\to p$.

② $\lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right)$ means taking limit with $x\to p$ first, followed by taking limit with $y\to q$.

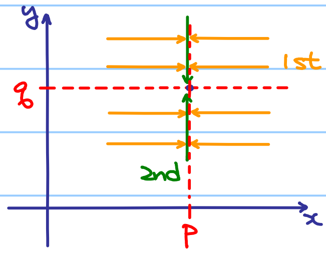
③ $\lim_{(x,y)\to(p,q)} f(x,y)$ is considering the behaviour of $f(x,y)$ when (x,y) tends to (p,q).

(in any arbitrary way)

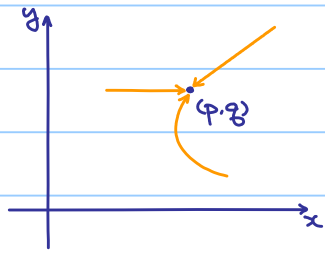
Question: $\lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right) = \lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right) = \lim_{(x,y)\to(p,q)} f(x,y)$?



$\lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right)$



$\lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right)$



$\lim_{(x,y)\to(p,q)} f(x,y)$

Example 6.13

Let $f(x,y) = \dfrac{x+y}{x-y}$.

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x+y}{x-y}\right) \overset{\text{fix } x}{=} \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x+y}{x-y}\right) \overset{\text{fix } y}{=} \lim_{y\to 0} \frac{y}{-y} = \lim_{y\to 0} -1 = -1$$

From exercise 6.3, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Exercise 6.4

Let $f(x,y) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{otherwise} \end{cases}$. Show that

a) $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$.

(Hint: Show that $\lim_{y\to 0} f(x,y) = \lim_{x\to 0} f(x,y) = 0$.)

b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(Hint: Consider $\gamma_1(t) = (t,0)$ and $\gamma_2(t) = (t,t)$ and show $\lim_{t\to 0} f(\gamma_1(t)) \neq \lim_{t\to 0} f(\gamma_2(t))$.)

$\therefore \lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right) = \lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right) \not\Rightarrow \lim_{(x,y)\to(p,q)} f(x,y)$ exists.

Exercise 6.5

Let $f(x,y) = \begin{cases} x\cos\left(\frac{1}{y}\right) + y\cos\left(\frac{1}{x}\right) & \text{if } x, y \neq 0 \\ 0 & \text{if otherwise} \end{cases}$. Show that

a) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(Hint: Use polar coordinates)

b) $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ and $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$ do not exist.

(Hint: Show that $\lim_{y\to 0} f(x,y)$ does not exist for all $x \neq 0$ and

$\lim_{x\to 0} f(x,y)$ does not exist for all $y \neq 0$.)

$\therefore \lim_{(x,y)\to(p,q)} f(x,y)$ exists $\not\Rightarrow \lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right), \lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right)$ exist

A sufficient condition for $\lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right) = \lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right) = \lim_{(x,y)\to(p,q)} f(x,y)$ is given by

Theorem 6.2 (Moore-Osgood Theorem)

If $\lim_{x\to p} f(x,y)$ exists pointwise for any $y \neq q$ and if $\lim_{y\to q} f(x,y)$ converges uniformly for $x \neq p$,

then all $\lim_{x\to p}\left(\lim_{y\to q} f(x,y)\right)$, $\lim_{y\to q}\left(\lim_{x\to p} f(x,y)\right)$ and $\lim_{(x,y)\to(p,q)} f(x,y)$ exist and are equal.

§ 7 Continuity

Definition 7.1

Let $D \subseteq \mathbb{R}^n$ such that every point $\vec{x}_0 \in D$ is a cluster point of D (so it makes sense to talk about $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})$.), and let $f: D \to \mathbb{R}$.

f is said to be continuous at $\vec{x}_0$ if $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

Idea:

③ equal

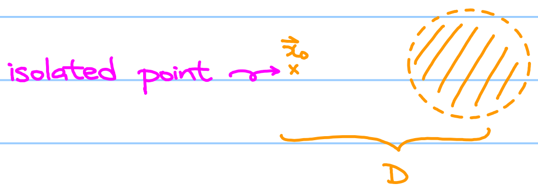
$$\underbrace{\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})}_{\text{① limit exists}} = \underbrace{f(\vec{x}_0)}_{\text{② well-defined}}$$

Technical issue: If $\vec{x}_0 \in D$ is not a cluster point of D, then there exists $r>0$ such that

$B_r^\circ(\vec{x}_0) \cap D = \phi$ ($\vec{x}_0$ is called an isolated point).

$f: D \to \mathbb{R}$ is always continuous at $\vec{x}_0$

(It can be seen clearly from ε-δ definition.)



Definition 7.2 (ε-δ definition)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in D$ and let $f: D \to \mathbb{R}$. f is said to be continuous at $\vec{x}_0$ if

for all $\varepsilon>0$, there exists $\delta>0$ such that $|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon$ for all $\vec{x} \in D$ with $|\vec{x}-\vec{x}_0| < \delta$ $^{+}$ ——— (*)

If f is continuous at every point in D, then f is said to be a continuous function.

Remark (+) · We can see that there is no restriction on $0<|\vec{x}-\vec{x}_0|$, i.e. $\vec{x} \neq \vec{x}_0$, it is simply because when $\vec{x}=\vec{x}_0$, $|f(\vec{x})-f(\vec{x}_0)| = 0 < \varepsilon$ and so the first inequality is satisfied automatically.

If $\vec{x}_0$ is a cluster point, (*) means $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$;

otherwise we can choose δ to be sufficiently small so that $\vec{x}=\vec{x}_0$ is the only point in D with $|\vec{x}-\vec{x}_0| < \delta$, (*) is satisfied automatically which means f is continuous at $\vec{x}_0$.

Example 7.1

Let $f(x,y) = \begin{cases} \dfrac{x^4 - y^4 + x^2 + y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) ; \\ 1 & \text{if } (x,y) = (0,0) . \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4 - y^4 + x^2 + y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2+1)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x^2 - y^2 + 1 = 1$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 1$ and $f(x,y)$ is continuous at $(x,y) = (0,0)$.

Theorem 7.1

Every elementary function (function constructed from constants, power functions, trigonometric, inverse trigonometric, exponential and logarithmic functions, via addition, subtraction, multiplication, division and composition) is continuous at all points at every point of its domain.

Example 7.2

- $f(x,y,z) = x^3 + 2xy^2 - 3yz + 2z + 5$ is continuous at every point in $\mathbb{R}^3$.
- $f(x,y) = e^{\sin(x+y)}$ is continuous at every point in $\mathbb{R}^2$.

  In particular, f is continuous at $(0,0)$ and so $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

  $$\lim_{(x,y)\to(0,0)} e^{\sin(x+y)} = e^{\sin(0+0)} = e^0 = 1$$

- $f(x,y) = \dfrac{x^4 - y^4 + x^2 + y^2}{x^2+y^2}$ is continuous at every point in $\mathbb{R}^2$ except $(0,0)$
- $f(x,y) = \ln(1+x+y)$ is continuous at every point $(x,y) \in \mathbb{R}^2$ with $1+x+y > 0$.

Theorem 7.2 (Extreme Value Theorem)

Let K be a compact subset of $\mathbb{R}^n$ and let $f: K \to \mathbb{R}$ be a continuous function

Then there exist $x_m, x_M \in K$ such that $f(x_m) \le f(x) \le f(x_M)$ for all $x \in K$, i.e. f attains absolute minimum and maximum at some points $x_m, x_M \in K$.

Remark: Note that $[a,b]$ is a compact set, if $K = [a,b] \subseteq \mathbb{R}$, the above theorem is just the extreme value theorem in single variable calculus

Example 7.3

Let $S^2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

Let $f: S^2 \to \mathbb{R}$ be a function defined by $f(x,y,z) = x+y+z$.

Exercise : Show that f is a continuous function, i.e. $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$ for all $\vec{x}_0 \in S^2$.

Furthermore, since S^2 is compact, by the extreme value theorem,

f attains absolute minimum and maximum at some points $x_m, x_M \in S^2$.

Question : Well, but how do we find ?

Proposition 7.1

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $D \subseteq \mathbb{R}^n$.

Then, the restriction of f on D, $f|_D : D \to \mathbb{R}$ defined by $f|_D(\vec{x}) := f(\vec{x})$ for all $\vec{x}\in D$,

is also a continuous function.

Example 7.3 (Continue)

Let $S^2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

Note that $f: \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) = x+y+z$ is a continuous function,

so $f|_{S^2} : S^2 \to \mathbb{R}$ is a continuous function.

Definition 7.3 (ε-δ definition)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in D$ and let $f: D \to \mathbb{R}^m$. f is said to be continuous at $\vec{x}_0$ if

for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon$ for all $\vec{x} \in D$ with $|\vec{x}-\vec{x}_0| < \delta$ —— (*)

However, if we write $f(\vec{x}) = f(x_1,x_2,\cdots,x_n) = (f_1(x_1,x_2,\cdots,x_n), f_2(x_1,x_2,\cdots,x_n), \cdots, f_m(x_1,x_2,\cdots,x_n))$, we have :

Proposition 7.2

f is continuous at $\vec{x}_0$ if and only if $f_i(\vec{x})$ is continuous at $\vec{x}_0$ for $i=1,2,\cdots,m$

proof :

It suffices to consider the case that $\vec{x}_0$ is a cluster point of D.

f is continuous at $\vec{x}_0 \Leftrightarrow \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

i.e. $\lim_{\vec{x}\to\vec{x}_0} (f_1(\vec{x}), f_2(\vec{x}), \cdots, f_m(\vec{x})) = (f_1(\vec{x}_0), f_2(\vec{x}_0), \cdots, f_m(\vec{x}_0))$

$\Leftrightarrow \lim_{\vec{x}\to\vec{x}_0} f_i(\vec{x}) = f_i(\vec{x}_0)$ for $i=1,2,\cdots,m$.

$\Leftrightarrow f_i(\vec{x})$ is continuous at $\vec{x}_0$ for $i=1,2,\cdots,m$